"Sets of Operations in Relation to Groups of Finite Order." By
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(Abstract.)

Introduction.

The present paper is concerned with the Theory of Groups of Finite Orders. The more general object of the paper is to place this theory in relation to a special algebra of the type considered in the general theory of Universal Algebra. This special algebra, which may be called the Algebra of Groups of Finite Order, has many affinities to the Algebra of Symbolic Logic; and a comparison of it with this algebra is given in the last section of this paper.

Mathematicians are accustomed in the study of quaternions to the idea of a vector symbol being considered from two points of view according to circumstances, namely, either as a geometrical entity, or as a symbol expressive of the operation of modifying some geometrical entity into another geometrical entity. Now from the point of view of this paper it is natural to abandon the idea of a group of N operations $S_0, S_1, \ldots\ldots S_{N-1}$ on some unspecified object, as being an idea which, however vaguely, appertains to a special interpretation of the symbols. The N symbols $S_0, S_1, \ldots\ldots S_{N-1}$ are to be considered, as in the similar case of quaternions, primarily as N distinct objects. When two of these objects are multiplied, as in $S_q S_r$, then a third object of the group, such as S_p, is produced; and in reference to this multiplication one of the symbols, say S_q, may be looked on as an operation on S_r modifying it into S_p. But this is not the sense in which the symbols are usually called operations in the Theory of Groups. However, in order not to disturb the well understood nomenclature of the subject, the N objects $S_0, S_1, \ldots\ldots S_{N-1}$ will always be called the fundamental operations, or, more shortly, the operations. But the word operation can simply be regarded as a name for the objects represented by these N symbols.

These N symbols are considered to be capable of addition according to the law

$$S+S = S.$$

This is the well known law of addition in Symbolic Logic, and the introduction of numerical symbols as factors is thereby avoided.

The sum of a selection of the N fundamental operations, such as $S_p + S_q + S_r + S_t$, is called a set. If a set obeys certain special conditions it is called a group. The sum of the whole number (N) of fundamental

operations, namely, $S_0 + S_1 + \ldots\ldots + S_{N-1}$, obeys these conditions. This sum is called the complete group, and all other groups are its sub-groups.

The first six sections of this paper are devoted to the detailed establishment of this purely algebraic view of the subject. At times the modification in treatment from that adopted in the standard treatises on the subject, such as Burnside's 'Theory of Groups of Finite Orders,' is slight. Where the modification would be of no sufficient interest it has been simply omitted, and the theorems when wanted have been assumed as part of the general knowledge of the subject. Only so much reasoning has been given as will establish the principles of the Algebra of Groups of Finite Order, viewed as an algebra independent of any interpretation, however vague.

The more special object of this paper follows directly from the changed point of view from which the Theory of Groups is here regarded. The idea of the group is no longer so absorbing; the set takes its place as the fundamental general entity which has to be investigated. A group is a special type of set. Accordingly in this paper some of the general properties of sets are investigated. A set of operations has numerous groups associated with it, and these groups have many relations with each other which this paper cannot pretend to have exhausted. The fundamental idea of this part of the paper (*cf.* § 7) is the formation from a set H of an unending series of other sets, here called the successive powers of H, and in the notation of the algebra written H^2, H^3,...... This series is called the power sequence of H. Any group which contains H also contains its power sequence. The power sequence is proved to have a periodic property (*cf.* § 9) which introduces a curious analogy to recurring decimals. This periodicity is the foundation of the rest of the paper. It governs the relations to each other of the various allied groups and sets. The periodicity is expressed by an equation of the form

$$H^{n+sm+q} = H^{n+q},$$

where m is called the period of H, and n the characteristic, and s and q are any integers including zero. The number of theorems relating to m is very large.

"Note on the Enhanced Lines in the Spectrum of α Cygni." By Sir NORMAN LOCKYER, K.C.B., F.R.S. Received January 20 —Read February 2, 1899.

(PLATE 6.)

When engaged in the classification of stars, according to their photographic spectra, in 1893* I came across two sets of lines of

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